MRST Formalism Applied to the Elastic Scattering at Oblique Incidence by a Fluid-Filled Cylindrical Borehole

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\[
\left[ A^P \right]_n = \frac{1}{K_T} \sum_{n=-\infty}^{\infty} \begin{bmatrix} jKb^n_{PT} H_n^{(1)}(Kr) \\ Kb^n_{PT} H_n^{(1)}(Kr) \\ K_z(b^n_{PT} + b^n_{PT}^{PT}) H_n^{(1)}(Kr) \end{bmatrix} \Omega_n(z, t),
\]

(5)

where the unknown non-normalized scattering amplitudes \( b^n_{PQ} \) in the scattering channel \( Q \) \((Q=L, T_1, \) or \(T_2)\) are calculated from the boundary equations at \( r=a \):

\[
\begin{align*}
\sigma_{rr} & = 0, \\
u_r & = 0, \\
\sigma_r & = 0, \\
\sigma_z & = 0.
\end{align*}
\]

(6)

written for each type of incident polarization \( P \). In this last relations, \( \sigma_{rr} \) is the normal stress, \( \sigma_r \) and \( \sigma_z \) are tangential stresses, \( p = j^{-1} \sum_{n=-\infty}^{\infty} c_n^P J_n(Kzr)\Delta_n(z, t) \)

(7)

is the fluid pressure \((K_L = K_f \cos \alpha_f, K_T = \omega/c_f)\) and \( u_r \) the radial displacement. Subscripts \( s \) and \( f \) stand respectively for “solid” and “fluid”. The expressions of the scattering amplitudes are given in the Annex as functions of determinants obtained from the matricial form of Eqs. (6).

## 2 Scattering Matrix

### 2.1 Building up \( S \)

For a given mode \( n \), the \( 3 \times 3 \) scattering \( S \) matrix

\[
\begin{bmatrix} S^{PQ} \end{bmatrix}_{P, Q=L, T_1, T_2}
\]

(7)

is built up from the scattering amplitudes \( b^n_{PQ} \)

\[
S = I + 2 \left[ \chi^{PQ} b^n_{PQ} \right].
\]

(8)

Here, \( I \) is the identity matrix, and \( \chi^{PQ} \) are normalization coefficients ensuring the unitarity of \( S \) \((SS^T = S^TS = I)\) as well as its symmetry \((\chi^{OP} b^{OP} = \chi^{PO} b^{PO})\). Values of these coefficients are

\[
\begin{align*}
\chi^{PP} & = 1, \quad (P = L, T_1, T_2) \\
\chi^{LT_1} & = X/X_T = (X_t^{LT_1})^{-1}, \\
\chi^{LT_2} & = X/X_z = (X_t^{LT_2})^{-1}, \\
\chi^{TT_2} & = X_T/X_z = (X_t^{TT_2})^{-1},
\end{align*}
\]

(9)

given the normalized frequencies \( X = Ka, X_T = K_T a \) and \( X_z = K_z a \).

### 2.2 Elastic pole contributions in \( S \)

From now on, a water-filled cavity inside aluminium medium will be considered \((\rho = 2790 \text{ kg.m}^{-3}, c_L = 6380 \text{ m.s}^{-1}, c_T = 3100 \text{ m.s}^{-1}, \rho_f = 1000 \text{ kg.m}^{-3}, c_f = 1470 \text{ m.s}^{-1})\). moduli of the \( S \) components plotted versus normalized frequency \( X_L = K_L a \) and longitudinal incidence angle \( \alpha_L \) exhibit a background amplitude perturbed by sharp peaks (see Figure 1).

![Figure 1](image_url)

Figure 1: Plots of \( |S_{LT_1}| \), \( |S_{LT_2}| \) and \( |S_{TT_2}| \) in the \((X_L, \alpha_L)\) plane for mode \( n=5 \).

As in normal incidence \([1-2]\), these peaks are due to the resonances of the fluid cylinder. Their frequencies \(X_t^{(*)}\) are approximately given by the real part \( X_t^{(r)} \) of the \( S \) poles \( X_p = X_t^{(r)} + jX_t^{(s)} \). These poles are close to
the roots $X_{0\ell}$ of the free mode equation $J_n(X_{\perp}) = 0$.

In Figure 2, variations of $X'_{\rho}$ and $X_{0\ell}$ (organized in trajectories denoted by $F_{\ell}$, $\ell \in \mathbb{N}$) versus $\alpha_L$ are compared for mode $n = 5$, exhibiting weak dependency on this parameter. In the following, the Multichannel Resonant Scattering Theory (MRST) formalism will be implemented to separate these resonant contributions from the background scattering in $S$.

Figure 2: Free water cylinder modes (solid lines) compared to real parts $X'_{\rho}$ of related poles (squares) versus incidence angle $\alpha_L$ (mode $n=5$).

3 MRST

3.1 Background factorization

As a preliminary remark, it is useful to emphasize the relation

$$\mathbb{S} = (d_{11}d_{(\rho)}S^{(r)} - d_{21}d_{(s)}S^{(s)})/d$$  \hspace{1cm} (10)

where $\mathbb{S}$ is expressed as the combination of two scattering matrices defined as follows:

$$S^{(r)} = S(\rho_f \to \infty)$$  \hspace{1cm} (11)

is the unitary and symmetric scattering matrix of determinant $\det(S^{(r)}) = S^{(r)} = d^{(r)*}/d^{(r)}$ built up in a case of a rigid wall cavity,

$$S^{(s)} = S(\rho_f = 0)$$  \hspace{1cm} (12)

is the unitary and symmetric scattering matrix of determinant $\det(S^{(s)}) = S^{(s)} = d^{(s)*}/d^{(s)}$ corresponding to the case of a “soft” wall cavity. The other coefficients occurring in relation (10) are defined in the Annex. In relation (10), the term $d_{21}$ containing the free mode characteristic function of the fluid cylinder is weighted by $S^{(s)}$: this point clearly indicates that this $S^{(s)}$ can be used to remove the background from $S^{(s)}$ in order to isolate the resonances. From this sign, let define the unitary scattering matrix

$$S^{(r)} = S^{(s)\dagger}S$$  \hspace{1cm} (13)

and the associated transition matrix

$$T^{(r)} = [T^{(r)PQ}] = -j(S^{(r)}-I)/2.$$  \hspace{1cm} (14)

The modulus of $T^{(r)}$ components plotted in $(X_L,\alpha_L)$ plane (Figure 3) exhibit only resonance peaks located at the same frequencies than those noted on $\mathbb{S}$ components plot (Figure 1) suggesting at a first sight that $S^{(s)}$ is the suitable background scattering matrix. In order to definitely confirm this assumption, the eigenvalues of $S^{(r)}$ have to be evaluated.

Figure 3: Plots of $|T^{(r)\perp T_1}|$ (a), $|T^{(r)\perp T_2}|$ (b), $|T^{(r)\perp T_3}|$ (c), in the $(X_L,\alpha_L)$ plane for mode $n=5$. 

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3.2 \( S^(*) \) Eigenvalues

An elegant way to calculate the eigenvalues of \( S^(*) \) is to introduce expansion (10) in the unitarity relation 
\[
SS^\dagger = SS^\dagger = I .
\]
As \( S^(*) \) and \( S^s \) are also unitary, it follows that
\[
qS^{s\dagger}S^s + qS^{s\dagger^2}S^s = I ,
\]
with \( q = d^{(r)}d^{(s)^*}/(d^{(r)}d^{(s)^*} + d^{(s)}d^{(s)^*}) \). From this, the eigenmatrix \( S^{s\dagger^2} \) of \( S^{s\dagger}S^s \) can be easily deduced by replacing this last term and its adjoint in (15) by the change of basis relation
\[
S^{s\dagger}S^s = R S^{s\dagger^2} R^\dagger ,
\]
where \( R \) is a rotation matrix which components will be studied in the next section. We found
\[
S^{s\dagger^2} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & S^{s}\end{bmatrix} .
\]

It results from (13) and (10) that
\[
S^(*) = R(\frac{d^{(r)}S^{s\dagger^2}S^s - d^{(r)}d^{(s)}I}{d})R^\dagger ,
\]
providing obviously the relation between \( S^(*) \) and its eigenmatrix \( S^{s\dagger^2} \). Then, \( S^{s\dagger^2} \) has necessarily the same structure as \( S^{s\dagger^2} \)
\[
S^{s\dagger^2} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & S^{s}\end{bmatrix} .
\]

where the two first eigenvalues equal to 1 indicate two closed eigenchannels. The third eigenvalue
\[
\det(S^*) = \left(\frac{d^{(s)} / d^{(s)^*}}{d^{(r)}d^{(s)^*} - d^{(r)^2}}\right) / d
\]
contains all the resonant interaction with the fluid cavity, as attested by the plot of the corresponding transition amplitude \( T^* = -jS^* - 1/2 \), exhibiting resonant shapes with unitary amplitudes and centred close to each zeros \( X_{0f} \) of \( J^r_n(X) \) (See Figure 4).

Indeed, as \( T^* \) can be expressed as a function of determinants
\[
T^* = \frac{d^{(r)}d^{(s)^*} - d^{(r)^2}}{d^{(r)}d^{(s)^*} - d^{(r)^2}} \]
asymptotic expansions of \( d^{(r)} \) and \( d^{(s)} \) for \( X_L >> n \) provide a quasi imaginary value of the ratio \( d^{(r)} / d^{(s)} \). Then, the Taylor expansions of \( d^{(r)} \) and \( d^{(s)} \) at the vicinity of \( X_{0f} \) allow to approximate \( T^* \) as a typical Breit-Wigner function
\[
T^* \approx \frac{-\Gamma f / 2}{X_L - X_{0f} + j\Gamma f / 2}
\]
with resonance frequency and width approximately given by
\[
\Gamma f / 2 = \frac{\rho_f c_f^2}{\rho_s c_L} \left( \frac{1}{\cos^2 \alpha_f} \right) \left[ \frac{X_L^2 \Im \left( \frac{d^{(r)}}{d^{(s)}} \right) }{X_L^2 d^{(s)}} \right] X_{0f},
\]
\[
X_{0f} = \frac{c_f^2}{\rho_s c_L \cos \alpha_f} \left[ n + \ell + \frac{1}{2} \right] \frac{\pi}{2}.
\]

These asymptotic values are compatibles with those found in normal incidence [1]. Relation (22) definitely demonstrates that \( S^(*) \) is a purely resonant scattering matrix and consequently that \( S^s \) is the adapted background to be removed from \( S \) in order to isolate the fluid resonances.

Figure 4: Plot of \( |T^*| \) versus normalized frequency \( n=5, \alpha_L=50^\circ \).

3.3 Density Matrix

The change of basis between \( S^(*) \) ad \( S^{s\dagger^2} \) is ensured by the same rotation matrix

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & S^{s}\end{bmatrix}
\]
that diagonalizes the product $S^{(*)}S^{(*)}$, as shown in relation (18)

$$S^{(*)} = \mathbb{R} S^{(*)} \mathbb{R}^\dagger.$$  

By introducing relation (14) in this last change of basis relation, the spreading of the fluid cylinder resonances over the scattering channels is fully and conveniently characterized by the density matrix $\rho$ such that

$$S^{(*)} = \mathbb{I} + 2j \rho S^{(*)}.$$  

This matrix defined as

$$\rho = \mathbb{R} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbb{R}^\dagger,$$

only depends on the normalized eigenvector associated to $S^{(*)}$, i.e. the third column $[\eta_3 \ r_{23} \ r_{33}]$ of $\mathbb{R}$.

An interesting way to understand the meaning of $\rho$, consists in emphasizing its role in relation (16):

$$S^{(*)\dagger} S^{(*)} = \mathbb{I} + 2j \rho T^{\mathrm{eig}}$$

with the transition amplitude associated to $T^{\mathrm{eig}} = -j(S^{(*)\dagger} S^{(*)} - 1)/2$. It results that the coupling

$$\rho = \frac{S^{(*)\dagger} S^{(*)} - S^{(*)}}{S^{(*)\dagger} S^{(*)} - S^{(*)}}$$

of fluid resonances in the scattering channel of $S^{(*)}$ depends on the “distance” between rigid and soft backgrounds. As a proof, the moduli of $T^{(*)}$ components are compared in Figure 5 to the corresponding $\rho$ components calculated by mean of relation (28).

In nuclear and particle physics [4], $\rho$ components are often expressed using mixing angles. Indeed, as the components of the third column of the rotation matrix $\mathbb{R}$ can be written

$$\begin{align*}
\eta_3 &= -\sin \beta \cos \varepsilon e^{-j(2\eta - \gamma - \tau)} \\
r_{23} &= -\sin \beta \sin \varepsilon e^{-j(2\eta - \gamma + \tau)} \\
r_{33} &= \cos \beta e^{j2\eta}
\end{align*}$$

the coupling angles $\beta$, $\varepsilon$, and the phase lags $\eta$, $\gamma$ and $\tau$ allow to define all components of $\mathbb{R}$.

Expressing now the modulus of each of these components in terms of partial widths

$$\Gamma_{LL} = \Gamma_{TT} = \Gamma_{TT} = \beta \sin^2 \varepsilon,$$

the resonant energy conservation is then merely provided by the trace of $\rho$

$$\text{Tr}(\rho) = \frac{\Gamma_{LL}}{\Gamma} + \frac{\Gamma_{TT}}{\Gamma} + \frac{\Gamma_{TT}}{\Gamma} = 1.$$  

Figure 5: Plots of $|T^{(*)} T|_{Q=LT1,T_2}$ and their envelop $|\rho^{T\Omega}|$ versus normalized frequency $X_L$ ($n=5$, $\alpha_L=50^\circ$).

4 Summary

The results presented in this paper have allowed us to extend from normal to oblique incidence previous studies performed on a fluid-filled cavity in a solid.
Using the factorization $S^{(n)} = S^{(n)} \dagger S$ for each $n$ mode, the resonances of the fluid cylinder are isolated in $S^{(n)}$, while other non resonant behaviour is confined in $S^{(n)}$. The calculus of $S^{(n)}$ eigenvalues and eigenvectors allow to predict the amount of resonant energy scattered in each outgoing channel and connect these quantities with the scattering by soft and rigid empty cavities.

As a final remark, the order of the previous processing performed on $S$, that is

(i) remove a background contribution from $S$,
(ii) calculate eigenmatrix and density matrix associated to $S^{(n)}$,

cannot be exchanged. Indeed, calculating first eigenmatrix $S_{\text{eig}}$ of $S$, provide a change of basis relation

$$ S = R_1 S_{\text{eig}} R_1^\dagger, \quad (34) $$

where all the eigenchannels are open and a part of the resonant energy of the fluid contributes to the components of the rotation matrix $R_1$. From this starting point, it is difficult, and even impossible, to isolate fluid resonances.

5 Annex

For each incident polarization $P$, the boundary equations (6) written at $r = a$ give rise to the linear system $D \begin{bmatrix} B^P \\ N^P \end{bmatrix} = 0$ where the vector

$$ B^P = \begin{bmatrix} c_n^P b_n^P b_n^{\dagger T_i} b_n^{P T_2} \end{bmatrix} $$

contains the unknown scattered amplitudes, and $N^P$ the incident contributions. The components of the $4 \times 4$ matrix $D = [d_{ij}]_{i,j=1\rightarrow 4}$ are

$$ d_{11} = \rho_F X_I^2 J_n(X), \quad d_{12} = (X_I^2 - X_2^2 - 2n^2) H_{n}^{(1)}(X) + 2X_H H_{n}^{(1)}(X), \quad d_{13} = 2n(X H_{n}^{(1)}(X) - H_{n}^{(1)}(X)), \quad d_{14} = -2(X H_{n}^{(1)}(X) + (X^2 - n^2) H_{n}^{(1)}(X)),$$

$$ d_{21} = -X_J J_n(X), \quad d_{22} = -X_H H_{n}^{(1)}(X), \quad d_{23} = n H_{n}^{(1)}(X), \quad d_{24} = X H_{n}^{(1)}(X),$$

$$ d_{31} = 0, \quad d_{32} = 2n(X_H H_{n}^{(1)}(X_H) - H_{n}^{(1)}(X_H)), \quad d_{33} = ((X^2 - 2n^2) H_{n}^{(1)}(X) + 2X H_{n}^{(1)}(X)), \quad d_{34} = -2n(X H_{n}^{(1)}(X) - H_{n}^{(1)}(X)),$$

$$ d_{41} = 0, \quad d_{42} = 2X_H X_J H_{n}^{(1)}(X_H), \quad d_{43} = -X_J n H_{n}^{(1)}(X), \quad d_{44} = X(X^2 - X_2^2) H_{n}^{(1)}(X),$$

where $X_J = K_J a$ and $X_H = K_H a$. Denoting by $d = d_1 d_3 - d_1 d_3$ the determinant of $D$, and by $d_{(e)}$ and $d_{(s)}$ the determinants related to the rigid and soft cavity scattering respectively, the scattered amplitudes are easily obtained by writing

$$ h_n^{PQ} = \frac{d_{PQ}}{d} \left| \begin{array}{cccc} d_{11} & d_{12} & d_{13} & -d_{13}^* \\ d_{21} & d_{22} & d_{23} & -d_{23}^* \\ 0 & d_{32} & d_{33} & -d_{33}^* \\ 0 & d_{42} & d_{43} & -d_{43}^* \end{array} \right|. $$

where $d_{PQ}$ is the modified determinant of $D$ in which the components of the column associated to the outgoing $Q$ channel have been replaced by the opposite and conjugate column components associated to the outgoing $P$ channel, for instance,

$$ d_{T_2 T_2} \begin{bmatrix} d_{11} & d_{12} & d_{13} & -d_{13}^* \\ d_{21} & d_{22} & d_{23} & -d_{23}^* \\ 0 & d_{32} & d_{33} & -d_{33}^* \\ 0 & d_{42} & d_{43} & -d_{43}^* \end{bmatrix}. $$

References


