Discrete Huygens’ Modeling for the Characterization of a Sound Absorbing Medium

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Acoustical engineers have been favored using the electrical analogy to acoustical problems. Based on the equivalence analogy for the wave propagation between the electrical transmission-line and the sound field, the authors proposed the use of the discrete Huygens’ modeling for time-domain solution method, which is known as the transmission-line matrix method (TLM) in electromagnetic engineering. The propagation is simulated by the sequences of the transmission and scattering in Huygens’ principle. The theory and the demonstrated examples were presented in the references [1][2]. The present work is concerned with the modeling of the lossy field for the characterization of the sound absorbing materials. A lossy component is introduced to the sound-absorbing field to facilitate the energy consumption. The frequency characteristics of the absorption coefficient are considered for the normal and oblique incidence.

1 Introduction

Figure 1: normally incident wave to a sound-absorbing layer

Consider a sound wave normally incident to sound absorbing medium at \( x = 0 \) as illustrated in Figure 1. The two acoustic media are made of air and sound absorber, which have the propagation speed of sound \( c_1, c_2 \) and the density \( \rho_1, \rho_2 \) respectively. The incident sound striking the boundary is partly transmitted, and partly reflected the pressures of which are defined by \( p_i, p_t \) and \( p_r \). The pressure reflection coefficient \( R \) between the two media is the ratio of the reflected to the incident defined by \( R = \frac{p_r}{p_i} \). The amount of energy going into the transmission and the reflection depends on the absorbing medium’s acoustic properties, from which the absorption coefficient is characterized. It is defined by a ratio of the absorbed energy to the incident energy so that \( \alpha = 1 - \frac{|R|^2}{1 - \frac{p_r}{p_i}} \). These property or absorption coefficient depends on the frequency so that \( \alpha(f) = 1 - \frac{|P(f)|^2}{|P_0(f)|^2} \). For one-dimensional modeling the transmission line method has been used for the sound-absorbing problem, which is solved in the frequency domain. In this paper, we alternatively propose the use of the transmission-line matrix modeling (TLM), which provides the time-domain approach. As the TLM method is equivalent to the Huygens’ principle, it simulates the physical process of the wave propagation and the wave scattering. When the frequency characteristics are required, the Fourier transform is utilized. That is, the temporal responses \( p_i(t) \) and \( p_r(t) \) are transformed into frequency responses \( P_i(f) \) and \( P_r(f) \). The frequency characteristics of the absorption coefficient \( \alpha(f) \) can also be derived.

2 Wave propagation on the one-dimensional TLM model

2.1 One-dimensional TLM element – Scattering matrix and wave equation [3]

Figure 2: One-dimensional field model
A one-dimensional TLM element consists of two branches whose characteristic impedance is $Z_0 = \sqrt{L/C}$. The lumped circuit element and distributed element of two ports are depicted in Figure 2a. $L$ and $C$ are the equivalent inductance and capacitance for the unit length. The homogeneous sound field is described by a series of the TLM elements are shown in Figure 2b. At the node $j$ of the element and at time $t = j\Delta t$, the input impulses are $\phi_j^i$ and $\psi_j^i$, and the output (transmitted and reflected) impulses $\phi_j^{i+1}$ and $\psi_j^{i+1}$ would result at time $t = (k+1)\Delta t$ with the delay of $\Delta t$. $k$ is the integer number and the superscript indicates the port number.

The relations of the sequence is given by

$$
\begin{bmatrix}
\psi_j^i \\
\phi_j^i
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\phi_{j+1}^i \\
\phi_j^i
\end{bmatrix}
$$

(1)

where $\phi$ and $\psi$ are the velocity potentials for the input impulses and output impulses respectively. So that the velocity potential at $i$ is

$$
\phi_i^i = \phi_i^1 + \phi_i^2
$$

(2)

As the output impulses become the input impulses of the adjacent elements, the compatibility conditions for connection are

$$
\phi_{i+1}^i = \phi_{i+1}^1, \hspace{1cm} \phi_i^{i+1} = \psi_i^1
$$

(3)

From equations (1) to (3), one obtains the expression

$$
\phi_{i+1}^i = 2\phi_{i+1}^1 - \phi_{i+2}^1 + \phi_i^2
$$

(4)

This is a finite difference-time domain expression, which can be expanded in Taylor series about $\phi_i^1$ to give the differential expression

$$
\frac{\partial^2 \phi}{\partial t^2} - c_i^2 \frac{\partial^2 \phi}{\partial x^2} + 2\left[\frac{\Delta t^2}{4!} \left( \frac{\partial^4 \phi}{\partial t^4} c_i^4 \frac{\partial^4 \phi}{\partial x^4} \right) + \frac{\Delta x^4}{6!} \left( \frac{\partial^6 \phi}{\partial t^6} - c_i^6 \frac{\partial^6 \phi}{\partial x^6} + \cdots \right) \right] = 0
$$

(5)

where $c_i$ is the transmission speed in free space. $c_i = \Delta x / \Delta t$.

Equation (5) corresponds to the wave equation for $\phi$ with higher order error terms removed as a result of the discretization, that is

$$
\frac{\partial^2 \phi}{\partial t^2} - c_i^2 \frac{\partial^2 \phi}{\partial x^2} = 0
$$

(6)

In acoustical engineering, pressure $p$ is often used for the velocity potential $\phi$. The relation of $p$ to $\phi$ is

$$
p = \rho \frac{\partial \phi}{\partial t} = \rho \frac{\Delta \phi}{\Delta t} = \phi_{i+1}^1 - \phi_{i-1}^1. \quad \text{In the following, as we refer to both quantities in relative values, they are treated as similar entity.}
$$

2.2 One-dimensional TLM element for lossy and variable propagation velocity medium

The sound-absorbing field is a lossy field of variable propagation speed. It can be modeled by providing the additional capacitance and conductance for changing the propagation speed and adding the loss. That is, the third branch with the characteristic impedance $Z_3 = Z_0 / \eta$ is connected and the forth branch of infinite length with the characteristic impedance $Z_4 = Z_0 / \zeta$ is also connected, as illustrated in Figure 3a. Elements are connected in series in Fig.3b. $\eta$ and $\zeta$ are the parameters with which the propagation speed and damping are properly chosen.

The scattering matrix expression for the velocity potential at $i$ are given by

$$
\begin{bmatrix}
\psi_i^i \\
\phi_i^i
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{2 + \eta + \zeta} & -\frac{\eta - \zeta}{2} & \frac{2}{2 + \eta} \\
\frac{2}{2 + \eta} & \frac{2}{2 + \eta} & \frac{2}{2 + \eta}
\end{bmatrix}
\begin{bmatrix}
\phi_i^i \\
\phi_i^{i+1} \\
\phi_i^{i+2}
\end{bmatrix}
$$

(7)

The impulse $\phi_i^i$ scattered to the forth is absorbed into infinity and does not go back, and

$$
\phi_i^i = \left(1 - \frac{\zeta}{2 + \eta + \zeta} \right) \left( \frac{2}{2 + \eta} \phi_i^i + \frac{2\eta}{2 + \eta} \phi_i^{i+1} \right)
$$

(8)
the attenuation factor or the attenuation value per unit length is defined as
\[
\chi = \frac{\zeta}{2 + \eta + \zeta}
\] (9)

The reflected impulses become the input impulses to the adjacent elements so that the compatible conditions are
\[
\phi_i^{n+1} = \psi_i^n, \quad \phi_{i+1}^{n+1} = \psi_{i+1}^n
\] (10)

one obtains the expression similar to the equation (4) so that
\[
\left(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n\right) + \frac{\zeta}{2 + \eta}\left(\phi_i^n - \phi_{i-1}^n\right)
= \frac{2}{2 + \eta}\left(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n\right)
\] (11)

This is the finite difference-time domain expression, from which one has the differential expression removing higher order error terms
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{c_i^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\zeta}{c_i \Delta t} \frac{\partial \phi}{\partial t} = 0
\] (12)

Equation (12) can be modified to be
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{c_i^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\zeta_2}{c_i \Delta t} \frac{\partial \phi}{\partial t} = 0
\] (12)'

where \(\zeta_2 = \frac{2}{\sqrt{2 + \eta}}\) and the propagation speed is
\[c_2 = \sqrt{\frac{2}{\sqrt{2 + \eta} c_1}}, \quad c_1 \text{ is the propagation speed in free space.}

### 3 Sound wave traveling into the sound-absorbing Layer

\[\text{Figure 4: The simulation of wave penetration into sound absorbing layer (one-dimensional TLM model)}\]

A test simulation is made for the one-dimensional wave incident into a sound-absorbing layer from free space as shown in Figure 4, where the sound speed and the density of two media are chosen to be \(c_1 = 340m/s, c_2 = 240m/s\) and \(\rho_1 = 1.2kg/m^3, \rho_2 = 16kg/m^3\) respectively. The attenuation factor \(\chi\) in sound-absorbing layer is chosen to be 0.01667. The whole field consists of 200\(\Delta l\) (\(\Delta l = 0.85mm\), element length) in which 10\(\Delta l\) is allocated for the sound-absorbing layer. A sound source is emitted at the left end and the sound-absorbing layer is backed by the rigid wall. The incident and reflected waves observed at the point \(i = 80\) are shown in Figure 5.

\[\text{Figure 5: The incident and reflected waves observed at point } i = 80\]

The amplitude spectra of the incident and reflected waves Fourier transformed are shown in Figure 6. Both \(P_i(f)\) and \(P_r(f)\) are complex number and their magnitude are shown in the figures. The absorption coefficient is the one defined in terms of the energy, so that
\[
\alpha(f) = 1 - \left|\frac{P_i(f)}{P_r(f)}\right|^2
\] (13)

which is shown in figure 8.

\[\text{Figure 6: Fourier transformed from the temporal domain to frequency domain}\]
4 Propagation on one-dimensional transmission line

The transmission line theory is well established in electrical engineering in which the solution is given in frequency domain [4]. No direct time-domain solution has been attempted except that it has been obtained by the inverse Fourier transform from the frequency domain solution. A one-dimensional transmission line model is shown in the Figure 7. For figure (a), the wave equation is given for the voltage $\phi$ by

$$\frac{d^2\phi(x)}{dx^2} - LC\frac{d^2\phi(x)}{dt^2} = 0$$  (14)

where $L$ and $C$ are the inductance and capacitance per unit length. It is the same as the equation (6), in which propagation speed is $c_1 = 1/\sqrt{LC}$.

The solution is of the form

$$\phi(x) = \phi^+ e^{-j\beta x} + \phi^- e^{j\beta x}$$  (15)

where $\phi^+ e^{-j\beta x}$ is the wave propagating in the $+x$ direction, while $\phi^- e^{j\beta x}$ is the wave propagating in the $-x$ direction. The constant $\beta = \omega\sqrt{LC}$ is the phase constant.

For figure (b), the wave equation is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2}L_2C_2 \frac{\partial^2 \phi}{\partial t^2} - GL_2 \frac{\partial \phi}{\partial t} = 0$$  (16)

where $L_2$ and $C_2$ are the inductance and capacitance in lossy transmission line. It is the same as the equation (12)' but for the propagation speed $c_2 = 1/\sqrt{L_2C_2}$ and the shunt conductance $G = \zeta_2/\sqrt{L_2/C_2} = \zeta_2 / Z_2$, where $Z_2 = \sqrt{L_2/C_2}$ is the characteristic impedance of the transmission line. The solution for $\phi(x)$ is of the form

$$\phi(x) = \phi^+ e^{-\gamma x} + V^+ e^{\gamma x}$$  (17)

where $\gamma = j\beta + \alpha = \sqrt{j\omega L_2 (j\omega C_2 + G)}$. $\alpha$ is the attenuation constant.

Fig.7 (b) shows a lossy transmission line, in which the shunt conductance $G$ is inserted. The wave equation is

By comparing the equation (6), (14) and (12)', (16), one can establish the relation $L = \rho_1$, $C = 1/(\rho_1 c_1^2)$ and $L_2 = \rho_2$, $C_2 = 1/(\rho_2 c_2^2)$, and $G = \zeta_2 / Z_2 = \sqrt{2 + \eta^2} / (Z_2\Delta l)$. At the interface between the two media, the reflection factor is given by

$$R = \frac{\sqrt{joL_2 / (joC_2 + G)} - \sqrt{joL / joC}}{\sqrt{joL_2 / (joC_2 + G)} + \sqrt{joL / joC}}$$  (18)

and transmission factor is...
\[ T = 1 - R \]  \hspace{1cm} (19)

The matrix expression for the incident and reflected waves are given as
\[
\begin{bmatrix}
P_i(f) \\
P_r(f)
\end{bmatrix} = \frac{1}{T} \begin{bmatrix}
1 & R \\
1 & 0
\end{bmatrix} e^{-\tau} \begin{bmatrix}
P_i \\
P_r
\end{bmatrix}
\]  \hspace{1cm} (20)

where \( P_i \) is the pressure at the end of the layer. As the boundary condition is rigid, the incident and the reflected are the same so that
\[
P_r = P_i
\]  \hspace{1cm} (21)

The comparison of the absorption coefficient for two models is shown in the Figure 8. The two methods give the same result and thus the present time domain method using TLM is reasonable and justifiable.

5 Two-dimensional model for oblique incidence

The absorption coefficient of the absorbing materials is known depending on the angle of the incidence. Here, a two-dimensional TLM modeling is considered to simulate the oblique incidence. The lumped circuit element and the distributed element for free space are depicted in Figure 9(a). \( L \) and \( C \) are the equivalent inductance and capacitance. Applying the Kirchhoff’s voltage and current laws on the circuit element, the differential expression about velocity potential \( \phi \) is derived
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}
\]  \hspace{1cm} (22)

where \( c = c_0 \sqrt{\frac{\mu}{\varepsilon}} \). It should be noted that the propagation speed in the two-dimensional network is slower than that in free space by the factor \( \sqrt{2} \).

The scattering matrix expression for the impulses is given by
\[
\begin{bmatrix}
\psi_i^1 \\
\psi_i^2 \\
\psi_i^3 \\
\psi_i^4
\end{bmatrix}_{k+1} = \begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
2 & 1 & -1 & -1 \\
1 & 1 & 1 & -1
\end{bmatrix} \begin{bmatrix}
\phi_i^1 \\
\phi_i^2 \\
\phi_i^3 \\
\phi_i^4
\end{bmatrix}_k + \begin{bmatrix}
1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & 1/4 & 1/4
\end{bmatrix} \begin{bmatrix}
\psi_i^1 \\
\psi_i^2 \\
\psi_i^3 \\
\psi_i^4
\end{bmatrix}_k
\]  \hspace{1cm} (23)

in which \( i \phi_i^n (n = 1 \sim 4) \) is the input impulse on the branch \( i \) at time \( t = k\Delta t \) and \( i \psi_i^n \) is the scattered one at the next time step. And the potential at node \( i \) is evaluated by
\[
i \phi_i = \left( \frac{i \phi_i^1 + i \phi_i^2 + i \phi_i^3 + i \phi_i^4}{4} \right)^{1/2}
\]  \hspace{1cm} (24)

A lossy TLM element for the sound-absorbing field is obtained by increasing the capacitance for adjusting the propagation speed and introducing the conductance for the loss, which is shown in the Figure 9(b). The scattering matrix expression is given as
\[
\begin{bmatrix}
\psi_i^1 \\
\psi_i^2 \\
\psi_i^3 \\
\psi_i^4
\end{bmatrix}_{k+1} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
\phi_i^1 \\
\phi_i^2 \\
\phi_i^3 \\
\phi_i^4
\end{bmatrix}_k + \begin{bmatrix}
1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & 1/4 & 1/4
\end{bmatrix} \begin{bmatrix}
\psi_i^1 \\
\psi_i^2 \\
\psi_i^3 \\
\psi_i^4
\end{bmatrix}_k
\]  \hspace{1cm} (25)

and the nodal potential is
\[
i \phi_i = \left( \frac{i \phi_i^1 + i \phi_i^2 + i \phi_i^3 + i \phi_i^4}{4} \right)^{1/2}
\]

The attenuation factor is the attenuation per one unit length
\[
\chi = \frac{\zeta}{4 + \eta + \zeta}
\]  \hspace{1cm} (27)

Consider a sound wave incident to the surface of the sound-absorbing layer surface at an angle \( \theta \). The layer is again backed by the rigid wall. A simulated geometry is shown in Figure 10. The sound speed and the density are \( c_0 = 340m/s \), \( \rho_0 = 1.2kg/m^3 \) for air, and for the sound-absorbing layer, they are \( c_L = 240m/s \), \( \rho_L = 16kg/m^3 \). The attenuation coefficient \( \chi \) of the sound-absorbing layer is again
taken to be 0.01667. The whole field consists of $2000\Delta \times 2000\Delta$ mesh ($\Delta = 0.85\,\text{mm}$), the surrounding of which has the non-reflective termination. The sound-absorbing Layer has $10\Delta$ thickness. The incident and reflected waves are observed at the observation point. From these waveforms, the absorption coefficient is calculated for the different angles of incidence. Some results are shown in the Figure.11.

6 Concluding Remarks

We proposed a lossy transmission-line matrix model (TLM) to characterize a sound-absorbing medium with which the time-domain solution is directly obtained. The Fourier transformed frequency characteristics are compared with the frequency domain solution based on the transmission-line model. Good agreement is obtained. Thus the relation between the physical parameters is established. The modeling is then extended to the two-dimensional case for the oblique incidence.

References


