Angle-of-Arrival Estimation for Acoustic Waves Propagating in Atmospheric Turbulence

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Atmospheric acoustic arrays are often used for localization of sound sources. However, common direction-finding algorithms do not directly compensate for the effects of an inhomogeneous propagation environment. In this paper we develop an angle-of-arrival estimator that accounts for the effects of atmospheric turbulence. The longitudinal coherence is needed for waves that arrive at oblique incidence to a planar array. Therefore, we also derive the longitudinal coherence of the sound field. The performance of the estimator is compared to the Cramer-Rao lower bounds.

1 Introduction

Acoustic sensor arrays are currently being used for source localization, detection, and identification in both atmospheric and underwater environments. However, the fluctuations in these propagation media may strongly distort the received acoustic signal. These distortions are perceived as fluctuations in the apparent angle of arrival (AOA) and source strength, and result in poor direction finding capabilities [1]-[4]. There currently exist many state-of-the-art direction-finding methods [5]-[9]. However, most methods either treat the propagation medium as homogeneous, or only account for the inhomogeneities of the medium through use of an effective sound speed. Song and Ritcey [10] were able to account for fluctuations in the ocean environment by developing a realistic, physics-based, statistical model of the received signal. From the statistical model, they were able to numerically calculate a maximum-likelihood estimator (MLE) of the AOA. Using a formalism similar to [10], Collier and Wilson [11]-[15] developed several physics-based, statistical models for acoustic signals propagating through atmospheric turbulence. These statistical models were used to calculate the performance bounds of acoustic arrays operating in different atmospheric conditions.

In this paper we use the monochromatic, plane-wave model of [13] to develop a MLE of the AOA for an acoustic wave propagating in atmospheric turbulence with fluctuations described by a von Kármán spectrum. This model assumes that the received signal may be modeled as a complex Gaussian random variable, whose first and second moments are determined from the theory of wave propagation in a random medium [16]. This model is reasonable for a signal that has been strongly diffracted and weakly or strongly scattered [17, 18]. Inherent in this model is the assumption that the magnitudes of the moments do not vary significantly from normal incidence to oblique incidence. By assuming this, the model is generally only valid for near normal incidence.

The approximation on the moments was made because the longitudinal coherence of the signal has not been previously determined. Therefore, we also derive an analytical solution for the longitudinal coherence.

The paper is structured as follows: in Sec. 2, the MLE is derived and results are presented for plane-wave propagation in a random medium with fluctuations described by von Kármán turbulence spectra. In Sec. 3, an integral equation is derived for the longitudinal coherence and an analytical solution is obtained for Gaussian turbulence spectra. Concluding remarks are given in Sec. 4.

Let $[\cdot]^\mathsf{T}$ denote the transpose, $[\cdot]^*$ the complex conjugate, $[\cdot]^\dagger$ the Hermitian adjoint, and $x \sim \mathcal{CN}_N(\mathbf{C}, \mu)$ denote that $x$ is distributed as an $N$-dimensional complex Gaussian random variable with mean $\mu$, $\mu_i = \langle x_i \rangle$, and covariance $\mathbf{C}$, $C_{ij} = \langle x_i x_j^* \rangle - \mu_i \mu_j^*$. 

2 MLE

In this paper, the model of [13], generalized to multiple snapshots, is employed for a monochromatic, plane-wave source propagating through atmospheric turbulence with fluctuations described by von Kármán spectra. We consider the single AOA estimation problem only, for wave propagation in a plane incident at a linear array with AOA $\phi$. A brief review of the statistical and physical models is given in Secs. 2.1 and 2.2, the estimation techniques con-
2.1 Statistical Signal Model

Suppose that we have an \(N\) element array. Let \(s(t_k) = p(t_k) + n(t_k) = [s_1(t_k), s_2(t_k), \ldots, s_N(t_k)]^T\) be the total received signal, where there are \(k = 1, \ldots, M\) uncorrelated snapshots. Here \(n \sim \mathcal{CN}_N(0, \sigma_n^2 I_N)\) is the noise and \(p \sim \mathcal{CN}_N(\mu_p, C_p)\) is the signal of interest, whose mean and covariance will be determined from the theory of wave propagation in a random medium. The dependence of \(C_p\) and \(\mu_p\) on the unknown AOA, \(\phi\), is implicit. We assume that the signal and noise are uncorrelated, so that \(s \sim \mathcal{CN}_N(\mu = \mu_p, C = C_p + \sigma_n^2 I_N)\). The probability likelihood function for \(M\) independent and identically distributed (IID) snapshots is

\[
\psi(s; \phi) = \frac{1}{\pi^M \det(C)} \exp \left\{ - \sum_{k=1}^M [s(t_k) - \mu]_i^\dagger C^{-1} [s(t_k) - \mu] \right\}
\]

(1)

The Cramer-Rao theorem states, that for any unbiased estimator \(\hat{\phi}\),

\[
\left\langle (\phi - \hat{\phi})^2 \right\rangle \geq J^{-1}(\phi)
\]

(2)

where \(J\) is the Fisher information (FI). The right-hand side of (2) is the Cramer-Rao lower bound (CRLB), which is the minimum mean-square error between the estimator \(\hat{\phi}\) and its actual value \(\phi\). The FI is

\[
J(\phi) = -\left\langle \frac{\partial^2 \ln \psi(s; \phi)}{\partial \phi^2} \right\rangle
\]

(3)

where the expectation is with respect to \(\phi\) and the derivative is evaluated at the true value of \(\phi\). The corresponding FI for (1) is [6]

\[
J = M \text{tr} \left[ C^{-1} \frac{\partial C}{\partial \phi} \right]^2 + 2M \text{Re} \left( \frac{\partial \mu_i}{\partial \phi} C^{-1} \frac{\partial \mu_i}{\partial \phi} \right)
\]

(4)

2.2 Physical Signal Model

We use the theory of wave propagation in a random medium, as derived by Ostahev [16], to determine the moments of \(p\). The analysis of [16] considers fluctuations in the medium velocity, and uses small-angle parabolic and Markov approximations. The ranges of applicability of these approximations are considered in detail in [16].

Consider a sound wave that is propagating with wavenumber \(k\), \(k = 2\pi/\lambda\), where \(\lambda\) is the wavelength. Let \(r\) be the propagation distance of the wavefront to the array center and let \(r_i\) be the vector from the array center to the \(i\)th sensor. Refer to Fig. 1. The first moment at the \(i\)th sensor, located at \(r_i\), is approximated to be

\[
\mu_i = \langle p_i \rangle = p_0 e^{i\Phi_i} e^{-\gamma r_i}
\]

(5)

where \(p_0\) is the pressure amplitude in a homogeneous medium, \(\gamma\) is the extinction coefficient for the first moment, \(\Phi_i = \chi + k \cdot r_i\) is the total phase at the \(i\)th sensor, and \(\chi\) is the phase at the array center. The second moment (also referred to as the coherence function) between sensors \(i\) and \(j\) is approximated to be

\[
\Gamma_{ij} \equiv \langle p_i p_j^\ast \rangle = p_0^2 e^{i\Phi_{ij}} e^{-\alpha(r_{ij})r_{ij}}
\]

(6)

where \(\Phi_{ij} = \Phi_i - \Phi_j = \chi + k \cdot r_{ij}\) is the difference in phases, \(r_{ij} = r_i - r_j = r_j' - r_i'\) is the separation vector between the sensors, and \(\alpha\) is the extinction coefficient of the second moment. For both (5) and (6), the first two terms represent the value in a homogeneous medium and the third term represents the decay due to scattering by fluctuations in the medium. At normal incidence, (5) and (6) correspond exactly to the results of [16]. For oblique incidence, (5) and (6) approximate that the magnitudes of the moments are given by their value at normal incidence. This approximation limits the applicability to near-normal incidence.

The extinction coefficients are dependent upon the structure of the random medium. For a plane wave

\[
\alpha(\rho) = 2\pi k^2 \left[ f(0) - f(\rho) \right]
\]

(7)

\[
\gamma = \alpha(\infty)/2 = \pi k^2 f(0)
\]

(8)

where \(f(\rho)\) is the 2D (or projected) correlation function for the sound-speed fluctuations. We consider the isotropic, homogeneous von Kármán model for the turbulence spectrum [16, 20]. For a vector field, induced by wind velocity fluctuations,

\[
\text{L 60}
\]
For this problem, we only estimate the single AOA, $\phi$. With an adjacent separation of $d$, is considered. For this problem, we only estimate the single AOA, $\phi$.

$$f_e (\rho, \varsigma^2, l) = \frac{2 \varsigma^2 l}{\sqrt{\pi} F(1/3)} \left( \frac{\rho}{2l} \right)^{5/6}$$

and for a scalar field, induced by temperature or humidity fluctuations,

$$f_s (\rho, \varsigma^2, l) = \frac{2 \varsigma^2 l}{\sqrt{\pi} F(1/3)} \left( \frac{\rho}{2l} \right)^{5/6} K_{5/6} \left( \frac{\rho}{2l} \right)$$

[16, 20]. Here, $\varsigma^2$ is the index-of-refraction variance, $l$ is the characteristic length scale of the turbulence, and $K_{5/6}$ is the modified Bessel function of order $\nu$.

Random motions in the atmosphere have characteristic time scales from seconds to minutes. For a time series of data that is less than the characteristic time scale, (1) should provide a reasonable, although idealized, representation. However, for a time series of data that spans over several characteristic time scales, a stochastic parametric model may better describe the statistics of the acoustic field. Detailed experimental analysis is needed to better understand the statistics of a sound field propagating in time-evolving atmospheric turbulence.

### 2.3 Estimation

The maximum likelihood estimator $\hat{\phi}$ is the estimator that maximizes $\varphi(\theta; \phi)$ in (1). There are several approaches to maximize $\varphi$ with respect to the unknown $\phi$. In maximizing $\varphi$, we are minimizing the log-likelihood function $\partial \ln \varphi / \partial \phi$. Two common numerical methods to minimize the log-likelihood function are Newton Raphson’s method and the method of scoring [6]. We have attempted both of these iterative methods. We found that the method of scoring failed to converge and Newton Raphson’s method converged to local extrema and not to the absolute minimum, even when using a close initial guess. As $\phi \in [0, \pi]$, we instead employ a grid search over this finite interval. We do this by successively using smaller and smaller intervals until we have determined the minimum of $\partial \ln \varphi / \partial \phi$ to within a given tolerance. This method works quickly for small values of $M$, but becomes cumbersome for larger values of $M$.

### 2.4 Results

We use simulated data to test the MLE. The signal and noise are individually sampled from complex Gaussian distributions. The AOA is then estimated assuming the correct distribution, and for mismatch assuming a homogeneous medium. A line array with 7 equally spaced microphones, with an adjacent separation of $d$, is considered. For this problem, we only estimate the single AOA, $\phi$.

Figure 2: Log-likelihood function in arbitrary units versus $\phi$. Considered here is sunny and windy conditions, $\sigma^2_r/p_0^2 = 0.1$, $\phi_{\text{act}} = 0$, $r/\lambda = 500$, and 10 IID samples. Similar results are found for the other atmospheric conditions.

All other parameters are assumed known. In [13], it was shown that the estimate of $\phi$ would decouple from the estimate of $\chi$, the phase at the array center, for this array geometry provided the origin was taken at the array center. For the estimations in this paper, it is assumed that the half space of the source is known, so that $\phi \in [0, \pi]$. Even if the half space is not known, then a planar array must be used to determine the AOA.

Consider four generic conditions: sunny and windy, sunny and calm, cloudy and windy, cloudy and calm. The coherence function is plotted in [11] as a function of sensor separation and range. The coherence is worst for sunny and windy conditions, which corresponds to the strongest turbulence intensity, and is best for cloudy and calm conditions. Large array apertures are often viewed as desirable to improve angle estimation for propagation in a homogeneous medium. One of the key results of [11]-[15], found by examining the CRLB of $\phi$ for various atmospheric conditions, is that the effects of turbulence produce a rolloff in the spatial coherence that ultimately degrades the performance despite the increasing aperture width.

Therefore, let us first examine the behavior of the log-likelihood function $\partial \ln \varphi / \partial \phi$ for different sensor separations. The log-likelihood function is plotted versus $\phi$ in Fig. 2 for normalized sensor separations of $d/\lambda = [0.01, 0.1, 0.5, 1.0, 1.5]$. Sunny and windy conditions are considered here for normal incidence ($\phi_{\text{act}} = 0$) at the line array. The normalized propagation distance is $r/\lambda = 500$, the noise-to-signal ratio is $\sigma^2_r/p_0^2 = 0.1$, and
10 IID samples are used. The value of $\phi$ that corresponds to the minimum of $\partial \ln \varphi / \partial \hat{\varphi}$ is the estimate $\hat{\varphi}$. We see that for $d/\lambda = 0.01$, the minimum is barely discernable, and for other values of the parameters $r$, $\sigma_2^2$, etc., the minimum cannot be obtained. This is an expected result, as there is little information available due to the sensor separation being only a very small fraction of the wavelength. For the “optimal” sensor separation of $d/\lambda = 0.5$, we see that the minimum is clearly discernable and the estimate of $\phi$ is accurate. However, for $d/\lambda = 1$, there are multiple minima with the same value. Therefore, we cannot estimate $\phi$ unless we a priori know that the wavefront is at near-normal incidence. We see that as $d/\lambda$ becomes increasing larger, the number of multiple, equal-valued, minima increases. Hence, this method of estimation can only be used when $0.1 \lesssim d/\lambda < 1$. Similar results apply for the other atmospheric conditions.

Figure 3 depicts $\Delta \phi$ the absolute value of difference between the estimate, $\hat{\varphi}$, and its actual value $\varphi_{\text{act}}$, for sunny and windy conditions. Here, we consider a noise-to-signal ratio of $\sigma_2^2/p_0^2 = 0.1$. The normalized propagation distance is $r/\lambda = 500$ and the normalized sensor separation is $d/\lambda = 0.5$. Depicted is the MLE resulting from correctly modeling the propagation medium (no mismatch shown by the blue dash-dotted line with square data points), the MLE resulting from incorrectly modeling the propagation medium as homogeneous (mismatch shown by the red dashed line with cross data points), and the result found from the Cramer-Rao lower bound. We see that in general the correctly modeled MLE outperforms the mismatch, in particular, for small values of $\varphi_{\text{act}}$, $-10^5 < \varphi_{\text{act}} < 10^5$. There appear to be some off-set resonances, in which the mismatch is a better estimate. Also note that the CRLB is the minimum value between the unbiased estimator and its actual value. Yet we see that the MLE is outperforming the CRLB in these resonances. These results are strongly dependent upon the sample considered, i.e., the seed of the random number generator. We are only considering $M = 10$ IID samples. Such a small sampling size may result in a poor representation of the Gaussian distribution. We have found larger, and even smaller, values of $\Delta \phi$ for different seeds. A better comparison of the MLE to the CRLB can be made by considering the ensemble average over numerous samplings. Further investigation is warranted, in particular, for different numbers of IID samples.

Figure 4 again depicts the same, for sunny and windy conditions, but for a noise-to-signal ratio of $\sigma_2^2/p_0^2 = 1.0$. Again we see similar results; however, in some regions the error is nearly half when correctly modeling the propagation environment.

Results for cloudy and calm conditions are shown in Figs. 5 and 6, for $\sigma_2^2/p_0^2 = 0.1$ and $\sigma_2^2/p_0^2 = 1.0$, respectively. Here we see similar results as for the sunny and windy conditions. However, the differences between

![Figure 3: $\Delta \phi = |\hat{\phi} - \phi|$ versus actual value of $\phi$. Sunny and windy conditions for $\sigma_2^2/p_0^2 = 0.1$, $\sigma_2^2/p_0^2 = 0.1$, $r/\lambda = 500$, $d/\lambda = 0.5$, and 10 IID samples.](image)

### 3 Longitudinal Coherence

As seen in the previous section, the coherence function $\Gamma$ of a sound field propagating in a turbulent atmosphere is an important statistical characteristic of this field. So far, only the transverse coherence function has been calculated and analyzed theoretically [16, 21], i.e., when the sensors are located in a plane perpendicular to the sound propagation path. When the sensors are not located in this plane, the longitudinal coherence function is needed.

#### 3.1 Derivation for a Plane Wave

Let $\mathbf{R} = (x, y, z)$ be the Cartesian coordinates and two points of observation be located at $(x', r_1)$ and $(x, r_2)$, see Fig. 7. Here, the $x$-axis is in the direction of predominant sound propagation, and $\mathbf{r} = (y, z)$ are the transverse coordinates. We assume that either a point source is located at the origin of the coordinate system or a plane sound wave is incident on a random medium located at $x \geq 0$.

The longitudinal coherence function of a sound field
and the Markov approximation, we obtain the longitudinal
small-angle approximation (\( r \ll L \)). Using the equation for
Fourier integrals with respect to \( \mathbf{r} \), the diagram technique,
the longitudinal coherence function may be expressed as

\[
\Gamma(x', x; \mathbf{r}_+ \mathbf{r}_-) = \int d^2 \kappa_+ e^{i \kappa_+ \mathbf{r}_+} \times \int d^2 \kappa_\perp e^{i \kappa_\perp \mathbf{r}_\perp} g(x', x; \kappa_+, \kappa_\perp) \tag{12}
\]

Here, \( g(x', x; \kappa_+, \kappa_\perp) \) is the spectral amplitude.

A closed formula for \( g(x', x; \kappa_+, \kappa_\perp) \) was derived in [22]
for the case of electromagnetic or sound wave propagating
in a medium with scalar random inhomogeneities. Using a one-way wave equation, the diagram technique, and the ladder approximation, \( g(x', x; \kappa_+, \kappa_\perp) \) was determined for \( x' > x \). For a plane sound wave propagating in
the direction of the \( x \)-axis, the transverse coherence function,
\( \Gamma_0(x; \mathbf{r}) \), does not depend on the coordinate \( r_+ \).

Using the equation for \( g(x', x; \kappa_+, \kappa_\perp) \) from [22], the small-angle approximation \( (r_+ \ll x, \ k, \kappa_\perp \ll k \) and the Markov approximation, we obtain the longitudinal
cohere function of a plane sound wave:

\[
\Gamma(x', x; \mathbf{r}) = \frac{k}{i 2 \pi (x'-x)} e^{i (\kappa - \gamma) (x'-x)} \times \int d^2 r_1 \exp \left( \frac{i k (r - r_1)^2}{2 (x' - x)} \right) \Gamma_0(x; \mathbf{r}_1) \tag{13}
\]

It follows from this formula that the longitudinal coherence function does not depend on the coordinates \( r_+ \).

For isotropic turbulence, the transverse coherence function,
\( \Gamma_0(x, \mathbf{r}) \), depends only on the modulus of the vector \( \mathbf{r} \). In this case, (13) may be directly integrated to obtain

\[
\Gamma(x', x; \mathbf{r}) = \frac{k}{i (x'-x)} \exp \left[ \frac{ik (r^2)}{2 (x'-x)} \right] \times \int_0^\infty dr_1 \Gamma_0(x; \mathbf{r}_1) \tag{14}
\]

for the longitudinal coherence function.

### 3.2 Gaussian spectrum

The transverse coherence function, \( \Gamma_0(x; \mathbf{r}) \), in (14) was calculated for the Kolmogorov, Gaussian, and von Karman spectra of temperature and wind velocity fluctuations [16]. It can be shown that, for the Kolmogorov and von Karman spectra, \( \Gamma_0(x, \mathbf{r}) \) is nonanalytic at \( r = 0 \). Therefore, let us consider Gaussian temperature and wind velocity spectra.

For the Gaussian spectra [16]

\[
\Gamma_0(x; \mathbf{r}) = I_0 \exp \left\{ -2 \gamma T x \left( 1 - e^{-r^2/L^2} \right) - 2 \gamma v x \left[ 1 - (1 - r^2/L^2)^{e^{-r^2/L^2}} \right] \right\} \tag{15}
\]

where \( I_0 \) is proportional to sound intensity, \( L \) is the outer
length-scale for the Gaussian spectra, and \( \gamma T \) and \( \gamma v \) are the extinction coefficients due to sound scattering by temperature and wind velocity fluctuations.
is determined as coherence function, given by
\[
r = \frac{1}{\gamma}\tan^{-1}\left(\frac{\gamma}{x'}\right).
\]
Note that, if \(\gamma\) is very small and, hence, can be equated to each other.

Substituting of (15) into (14) results in an integral which cannot be calculated analytically. Therefore, assume that \(r \ll L\). In this case, (15) can be written as:
\[
\Gamma_0(x; r) = I_0 \exp(-r^2/\gamma_0^2),
\]
where \(r_{c0}\) is the coherence radius of the transverse coherence function, given by
\[
r_{c0} = \frac{L}{\sqrt{2x(\gamma_T + 2\gamma_0)}} = \frac{L}{\sqrt{2x\gamma_T}}.
\]
where \(\eta = 1 + \gamma_0/(\gamma_T + \gamma_0), 1 \leq \eta \leq 2\).

Note that, if \(\gamma_T x \gg 1\) and \(\gamma_0 x \gg 1\), (16) correctly describes the coherence not only for small but also for large values of \(r\). Indeed in this case, it follows from (15) and (16) that, for \(r > L\), the coherence functions are both very small and, hence, can be equated to each other.

Substituting (16) into (14) and calculating the integral, we obtain the longitudinal coherence function
\[
\Gamma(x', x; r) = \frac{I_0}{\sqrt{1 + D_0^2}} \exp\left[\frac{(ik - \gamma)(x' - x) - i\phi}{\sqrt{1 + D_0^2}} + \frac{ikr^2}{2(x' - x)}\right] - \frac{r^2}{D_0^2} \right]^{-1/2}.
\]
Here \(D_0 = 2(x' - x)/(kr_{c0}^2)\) is the wave parameter with respect to the transverse coherence radius \(r_{c0}\), the angle \(\phi\) is determined as \(\tan \phi = 1/D_0^2\), and \(r_c\) is the coherence radius for the longitudinal coherence function given by
\[
r_c = r_{c0}\sqrt{1 + D_0^2}.
\]
This determines how \(r_c\) increases with increasing \(x' - x\).
Note that for \(x' = x\), (18) coincides with (16) as it should.

![Figure 6: Δφ = |φ' - φ| versus actual value of φ. Cloudy and calm conditions for \(σ_n^2/p_0^2 = 1.0\). Other parameters same as Fig. 3.](image)

![Figure 7: Coordinate system for longitudinal coherence calculations.](image)

Let us now study the longitudinal coherence when \(r = 0\), i.e., when the two points of observation are located along the sound propagation path. It follows from (18) that
\[
|\Gamma(x', x; 0)| = \frac{I_0}{\sqrt{1 + D_0^2}} \exp[-\gamma(x' - x)]
\]
\[
= \frac{I_0}{\sqrt{1 + D_0^2}} \exp[-\gamma(x' - x)]
\]
Here \(D = 4x/(kL^2)\) is the “classical” wave parameter (e.g., see (7.91) from [16]) which appears in the transverse statistical moments of the fields.

Figure 8 shows the dependence of \(|\Gamma(x', x; 0)|\) on the normalized distance \(\gamma(x' - x)\) between two points of observation for \(\eta = 1.5\) and different values of the wave parameter \(D\). It follows from Fig. 8 and (20) that the longitudinal coherence significantly depends on the parameter \(D\). Recall that small values of \(D\) correspond to the case when diffraction effects can be ignored when considering transverse statistical moments in the plane \(x = \text{const}\).

In this case, as one would expect, the longitudinal coherence is described by the term \(\exp(-\gamma(x' - x))\). The other limiting case is large values of \(D\) when diffraction effects are important. In this case the longitudinal coherence is much smaller, and is described by the square root in (18).

## 4 Conclusions

We have derived a MLE for the wavefront AOA that accounts for the effects of atmospheric turbulence with fluctuations described by von Kármán spectra. The physics-based, statistical model used is valid for strong diffraction and strong or weak scattering. The preliminary results indicate that the error bounds on the MLE are below the CRLB. As a result, more detailed analyses are needed, as a better comparison to the CRLB will result from ensemble averaging. In the MLE analyses, it was assumed that the magnitudes of the moments of the sound field could be approximated by their values at normal incidence. This limits the applicability of this model to near-normal incidence.

Therefore, we have also derived the longitudinal coher-
Figure 8: $|\Gamma(x', x; 0)|$ as a function of the normalized distance $\gamma(x' - x)$ between two points of observation for $\eta = 1.5$ and different values of the wave parameter $D$.

ence for a plane wave. Analytical solutions were not feasible for von Kármán spectra, therefore, we considered Gaussian spectra. It was found that for strong diffraction, the longitudinal coherence is much smaller than for weak diffraction, and decays by the square root in (18). This gives us an indication that the approximations used in the model for the MLE are reasonable, and perhaps not as limiting as initially believed. Further investigation of the longitudinal coherence for von Kármán spectra is needed.

References